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ON THE FORCE ACTING ON A BODY IN VISCOUS FLUID*

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Extension of the Kelvin–Tait formula to the case of a body moving in a viscous fluid is presented, and the problem of the analog of the Onsager reciprocity relations for media with memory is discussed.

The problem of determination of forces and moments acting on an absolutely rigid body in motion in an unbounded or bounded by fixed walls potential stream of perfect incompressible fluid was reduced by Kelvin and Tait /1–3/ to the problem of determining the kinetic energy of fluid when that energy is finite. The force and moment are determined in terms of the kinetic energy K by formulas

$$F_i = \frac{\partial K}{\partial r^i} - \frac{d}{dt}\frac{dK}{\partial r^i_{,t}}, \quad K = K(r^i, r^i_{,t}, \alpha_a{}^i, \alpha^i_{a,t}) \tag{1}$$
$$M_i = e_{ijk}\left(\frac{\partial K}{\partial \alpha_a{}^j} - \frac{d}{dt}\frac{\partial K}{\partial \alpha^j_{a,t}}\right)\alpha_a{}^k$$

where r^i are components of radius vector r of a fixed point of the body, $\alpha_a{}^i$ are components of the orthogonal matrix α which defines the position of unit vectors $(i, j, k, a, b, c = 1, 2, 3)$ accompanying the body, the comma preceding t in subscripts indicates differentiation with respect to t, and e_{ijk} are Levi-Civita symbols.

Similar formulas for a body in the Stokes flow of viscous fluid are linked with the dissipation potential of the fluid D by the expressions

$$F_i = -\frac{\partial D(r, \alpha; r_{,t}, \alpha_{,t})}{\partial r^i_{,t}} \tag{2}$$
$$M_i = -\frac{\partial D(r, \alpha; r_{,t}\alpha_{,t})}{\partial \alpha^j_{a,t}}\alpha^k_{a,t}e_{ijk}$$

In the case of Newtonian fluids D represents half of dissipation in the region of motion.

The question arises whether a universal relationship which would bind together the force, moment, kinetic energy and dissipation exist, when the dependence on not only instantaneous properties of motion but, also, on the motion previous history is available.

It is shown below that the variational equation

$$F_i\delta r^i + M_i\delta\varphi^i = \delta K - \frac{d}{dt}\delta_{.}K - \delta_{.}D \tag{3}$$

which holds at every instant of time t is an appropriate generalization of formulas (1) and (2) for a motion that begins from the state of rest. In this equation $\delta\varphi^i = {}^1/_2 e^{ijk}\alpha_j{}^a\delta\alpha_{ka}$ is the variation of the body angle of turn, and the kinetic energy and dissipation potential are some functionals of the history of motion. Their dependence on previous history can be defined by two groups of arguments (separated below by the semicolon)

$$K = \overset{\tau=t}{\underset{\tau=0}{K}}(r(\tau), \alpha(\tau);\ r_{,\tau}, \alpha_{,\tau})$$
$$D = \overset{\tau=t}{\underset{\tau=0}{D}}(r(\tau), \alpha(\tau); r_{,\tau}, \alpha_{,\tau})$$

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The method of separating the two groups of arguments is given below.

In (3) δ is the operator of variation with respect to both arguments, and $\delta_.$ is the operator whose action consists of varying with respect to the second group of arguments, with subsequent substitution of δr, $\delta\alpha$ for $\delta r_{,\tau}$, $\delta\alpha_{,\tau}$.

The following observations should be noted.

1°. Relations (1) and (2) follow from the variational equation (3), when K and D are functions of the instantaneous position and velocity of the body.

2°. The variational equation (3) shows that the six functionals of the history of motion F_i and M_i are expressed in terms of the two functionals K and D. Since Eq.(2) is a particular case of Eq.(3) and represents the Onsager principle for a slow motion of the body in a viscous fluid, Eq.(3) may be considered as the analog of Onsager's principle for interactions that are not localized with respect to time. In thermodynamic systems with neglibible inertial effects the respective generalization of the Onsager principle may be formulated as follows:

$$X_A \delta u^A = -\delta U + \frac{d}{dt}\delta_. U - \delta_. D$$

where X_A is the thermodynamic flux, du^A/dt are thermodynamic forces, and U and D are functionals of internal energy and dissipation, respectively.

3°. Setting in (3) for the instant of time t, $\delta r^i = \delta\varphi^i = 0$, we find that the functionals K and D must satisfy the condition: for any $\delta r^i(\tau)$ and $\delta\varphi^i(\tau)$ that vanish at $\tau = 0$ and $\tau = t$ the identity

$$\delta K - \frac{d}{dt}\delta_. K - \delta_. D = 0 \tag{4}$$

must be satisfied.

4°. The calculation of forces and moments acting on a solid body in a viscous fluid is extremely complicated. It is, hence, reasonable to use the variational equation (3) for semi-empirical determination of forces and moments, selecting functionals K and D so that they satisfy formula (4) and, then obtaining experimentally the free parameters.

Let us prove equality (3). For this we shall consider in some inertial system of coordinates x^i a vessel containing a homogeneous incompressible viscous fluid with body A in it. At the instant of time $t = 0$ the system is at rest

$$x^i(\xi^a, 0) = x_0^i(\xi^a), \quad v^i(\xi^a, 0) = 0, \quad r_{,t}(0) = 0, \quad \alpha_{,t}(0) = 0 \tag{5}$$

where ξ^a are Lagrangian coordinates of the fluid and solid body, $x^i(\xi^a, t)$ is the law of motion of the medium, and $v^i(\xi^a, t) = \partial x^i(\xi^a, t)/\partial t$ is the medium velocity.

Under the action of external forces the body begins to move in conformity with some law $r(t)$, $\alpha(t)$. We further assume that at $t = 0$:

$$r_{,tt}(0) = 0, \quad \alpha_{,tt}(0) = 0 \tag{6}$$

Motion of the fluid is determined by the equations of momenta and the continuity equation

$$\rho\frac{d^2x^i(\xi^a, t)}{dt^2} = -\nabla^i p + \mu\Delta v^i, \quad \det\left\|\frac{\partial x^i}{\partial\xi^a}\right\| = \det\left\|\frac{\partial x_0^i}{\partial\xi^a}\right\| \tag{7}$$

and by the boundary conditions at the boundary ∂A_ξ of the body and ∂V of the vessel

$$\begin{aligned} x^i(\xi^a, t) &= r^i(t) + \alpha_a^i(t)\xi^a, \quad \xi^a \in \partial A_\xi \\ x^i(\xi^a, t) &= x^i_*(\xi^a), \quad \xi^a \in \partial V \end{aligned} \tag{8}$$

In Lagrangian variables the body boundary ∂A_ξ is fixed relative to its particles, and $\xi^a = 0$ are Lagrangian coordinates of a point of the body at radius vector $x^i = r^i(t)$. It is assumed that in the considered here time interval the system of Eqs.(5)—(8) has a solution and that the solution is unique.

The law of fluid particle motion $x^i(\xi^a, t)$ and the pressure $p(\xi^a, t)$ can be considered at the instant of time t as functionals of the law of solid body motion $r(t)$, $\alpha(t)$. Let us assume that an infinitely small perturbation $\delta r(t)$, $\delta\alpha(t)$ of motion of the body takes place. Then $x^i(\xi^a, t)$ and $p(\xi^a, t)$ obtain some infinitely small increments δx^i and δp. The symbol δ denotes variations with constant ξ^a and symbol ∂ those with constant x^i. Equations for the determination of δx^i and δp obtained by varying Eqs.(7) are

$$\rho\frac{d^2\delta x^i}{dt^2} = -\nabla^i\delta p + \nabla_k p\nabla^i\delta x^k + \mu\Delta\frac{d\delta x^i}{dt} - \mu\nabla_k v^i\Delta\delta x^k - 2\mu\nabla_j\delta x^k\nabla^j\nabla_k v^i, \quad \nabla_i\delta x^i = 0 \tag{9}$$

Advantage is taken here of the property that variations δ with constant ξ^a are commutative with operator d/dt, and variations ∂ at constant x^i are commutative with operator ∇_i, and of the relation $\delta u = \partial u + \delta x^k\nabla_k u$.

By varying (5) and (8) we obtain for δx^i the initial and boundary conditions

$$\delta x^i = 0, \quad \frac{d\delta x^i}{dt} = 0 \quad \text{when} \quad t = 0 \tag{10}$$

$$\delta x^i = \delta r^i + \alpha_j^{\ a}(x^j - r^j)\delta\alpha_a^{\ i} \quad \text{on} \quad \partial A_\xi$$

$$\delta x^i = 0 \quad \text{on} \quad \partial V$$

In conformity with (9) and (10) the quantities δx^i are at instant of time t functionals of the previous history of motion $r(t)$, $\alpha(t)$ and of variations δr, $\delta\alpha$. We express this as follows:

$$\delta x^i = \mathop{l^i}_{\tau=0}^{\tau=t}(r(\tau), \alpha(\tau);\ \delta r(\tau), \delta\alpha(\tau)) \tag{11}$$

Although functionals (11) depend also on ξ, the latter is not shown among the arguments for brevity. As implied by system (9), (10) functionals (11) are linear with respect to δr, $\delta\alpha$ and, generally, nonlinear with respect to r, α.

Derivation of the variational equation (3) is based on the equality

$$v^i = \mathop{l^i}_{\tau=0}^{\tau=t}(r(\tau),\ \alpha(\tau);\ r_{,\tau}, \alpha_{,\tau}) \tag{12}$$

We begin the proof of relation (12) by stating that by virtue of assumption (6)

$$dv^i / dt = 0 \quad \text{when} \quad t = 0 \tag{13}$$

Indeed, by applying operation rot to the equations of momenta and taking into account that $v^i = 0$ at $t = 0$, we obtain

$$\operatorname{rot}(\partial v/\partial t) = 0 \quad \text{at} \quad t = 0$$

Moreover $\operatorname{div}(\partial v/\partial t) = 0$ at $t = 0$. On ∂V and ∂A_ξ we have $dv/dt = 0$ at $t = 0$. This is only possible when at $t = 0$ throughout the flow region $\partial v/\partial t = dv/dt = 0$.

We replace in (9) the symbol δ by the time derivative d/dt with constant ξ^a. Equations (9) are now transformed into Eqs.(7) differentiated with respect to time. We shall consider these equations as linear with respect to v^i and dp/dt as variable coefficients which we assume to be known functionals of $r(\tau)$, $\alpha(\tau)$.

At the initial instant of time v^i satisfy by virtue of (5) and (13) the conditions

$$v^i = dv^i/dt = 0 \quad \text{at} \quad t = 0$$

At the stream boundary, differentiating (8) with respect to time, we furthermore have

$$v^i = 0 \quad \text{on} \quad \partial V;\ v^i = r^i_{,t} + \alpha_j^{\ a}(x^j - r^j(t))\,\alpha_a^{\ i}{}_{,t} \quad \text{on} \quad \partial A_\xi$$

Thus the system of equations for the determination of v^i and dp/dt fully coincides with that for the determination of δx^i and δp and conforms to formula (12).

The kinetic energy of fluid and the dissipation potential are defined by formulas

$$K = \int_V \frac{1}{2}\rho v_i v^i\, dv, \quad D = \int_V \mu\nabla^i v^j\nabla_{(i}v_{j)}\, dv \tag{14}$$

where the parentheses in subscripts denote symmetrization.

The substitution of (12) into (14) determines the dependence of K and D on the two groups of arguments, and K and D are quadratic with respect to the second of these groups.

We multiply the equation of momenta by δx^i and integrate over the region occupied by the fluid. Using the boundary conditions for δx^i on ∂V and ∂A_ξ, and the definitions of force and momentum, we obtain

$$F_i\delta r^i + M_i\delta\varphi^i + \int_V 2\mu\nabla^{(i}v^{j)}\nabla_{(j}\delta x_{i)}\, dV + \int_V \rho\frac{dv^i}{dt}\delta x_i\, dV = 0 \tag{15}$$

where δx^i indicates the variation of $x^i(\xi^a, t)$ generated by the variation of $r(\tau)$ and $\alpha(\tau)$. Using the equality $\delta v^i = d\delta x^i/dt$ and formula (12) we rewrite (15) as

$$F_i\delta r^i + M_i\delta\varphi^i + \int_V 2\mu\nabla^{(i}v^{j)}\nabla_{(j}\overset{\tau=t}{\underset{\tau=0}{l_{i}}}(r(\tau),\alpha(\tau);\ \delta r(\tau),\delta\alpha(\tau))\,dv + \tag{16}$$
$$\frac{d}{dt}\int_V \rho v_i \overset{\tau=t}{\underset{\tau=0}{l^i}}(r(\tau),\alpha(\tau);\ \delta r(\tau),\delta\alpha(\tau))\,dv - \int_V \rho v_i\delta v^i\,dv = 0$$

According to (14) and (12)

$$\delta K = \int_V \rho v_i \delta v^i\,dV \tag{17}$$
$$\delta_{\cdot} K = \int_V \rho v_i \overset{\tau=t}{\underset{\tau=0}{l^i}}(r(\tau),\alpha(\tau);\ \delta r,\delta\alpha)\,dV$$
$$\delta_{\cdot} D = \int_V 2\mu\nabla^{(i}v^{j)}\nabla_i \overset{\tau=t}{\underset{\tau=0}{l_j}}(r(\tau),\alpha(\tau);\delta\tau,\delta\alpha)\,dV$$

From (16) and (17) follows (3).

For a deformable solid body defined by a finite number of degrees of freedom the variational equation (3) to which is added the work of generalized forces on the additional degrees of freedom, is also valid. Extension to Newtonian viscous compressible fluid does not present difficulties.

It is clear that the conclusion arrived at above and based on equality (12) are directly applicable to any system that is locally defined by equations of type (7), and after "averaging" (passing to a system with lesser number of degrees of freedom) local relations are replaced by variational realation (3).

It should be noted that the assumption of the motion starting from rest is essential. Otherwise (e.g., in the case of steady motion) additional terms appear in formula (12), and these will have to be taken into account in the variational equation (3).

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